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A Classification of the Possible Symmetry Groups of Liquid Crystals

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A classification of all the possible groups of rigid transformations, namely rotations, reflections, translations, screw axes and glide planes, is presented. These groups are the subgroups of the centrosymmetric Euclidean group. They may be used to describe the symmetry of crystals, liquids, amorphous solids, powders, and liquid crystals. They may also be used to describe the symmetry of systems of lower dimensionality, as surfaces, thin layers, etc.

1 INTRODUCTION

The phase transitions from liquids to crystals or to liquid crystals, are accompanied by breaking of the rotational and translational invariance of the liquids. The resulting crystal symmetries are the 230 space groups. These groups are characterized by the fact that each one of them contains a three-dimensional discrete translation subgroup, which is called a lattice. In the case of liquid crystals the translation subgroup needs not be discrete, but may be continuous or may contain continuous as well as discrete translations. The liquid crystals are, therefore, described by some subgroups of the full rotation-translation group, other than the space groups. A classification

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of all the possible symmetries of liquid crystals has, as yet, not been carried out. There exist, however, other classifications, the most important of which is that of Hermann.[1,2] He introduced four different types of statistical translation operators, and classified the possible three-dimensional structures, according to their invariance under these operators. Another classification is that of Baccara[3] who classified the liquid crystals according to their rotational (but not translational) symmetry.

In this work we present a classification of all the possible subgroups of the symmetry group, G_0, the centrosymmetric liquid.[4] Specifically, we are looking for all the groups of rigid transformations, namely, rotations, reflections, translations, screw axes, and glide planes. These groups may describe the symmetry of crystals, liquids, liquid-crystals, amorphous solids and powders, when they contain a three-dimensional translation subgroup. They may also describe the symmetry of surfaces, thin layers or molecules when the dimensionality of the translation subgroup is less than three. The question of how the symmetry of a given phase is defined is a most important but rather complicated one. In the case of crystals or centrosymmetric liquids one may define the symmetry as the symmetry of the thermodynamic average of the density function $\rho(r)$. However, this definition is inadequate for more complicated cases. For example, the density function $\rho(r)$ of nematic liquid crystals is an isotropic function, while the nematics are clearly uniaxial (or even biaxial). It was suggested by Landau,[5] that the symmetry of the nematics will be defined as the symmetry of their density–density correlation function $\rho_{12}(r_1, r_2)$. This definition is insufficient for other cases as the non-centrosymmetric liquids. In this case both $\rho(r)$ and $\rho_{12}(r_1, r_2)$ are invariant under inversion, while the liquid is not. In the present work, we shall make no attempt to present a general definition of the symmetry of a given phase. We shall identify the symmetry of the known liquid crystals by the symmetry group of the order parameter which is used to describe each phase. We hope to come back to the problem of the definition of the symmetry of a given phase in a later publication.

We found that the subgroups of G_0 may be classified into ten classes according to their translation subgroup (see Table IV). The groups which describe the symmetry of amorphous solids, liquids and liquid crystals belong to classes 1,2. We, therefore, list the point group and site symmetries of the groups of the first two classes. The physical interpretation of a site symmetry of amorphous solids is quite different from that of crystals, liquids or liquid crystals. In the latter case, the site symmetry of a special position is the symmetry of the time average of the "crystal field" at the special point. The time average must be taken over time intervals longer than the characteristic time of the thermal fluctuations. In the case of amorphous solids the time average of the "crystal field" at a special position has no symmetry, and the site symmetry is the symmetry of the "crystal field" averaged over different points of the same special position in the solid.

The symmetry group, G_0, of a centrosymmetric liquid, is the group which contains all possible rotations, reflections, and translations in a three-dimensional space. G_0 is the centrosymmetric Euclidean group,[6] and it can be written as:

$$G_0 = (R^3 \times I) \wedge P^3$$

where:

R^3—three dimensional rotation group
I—inversion
P^3—the continuous three-dimensional translation group
$A \times B$—the direct product of the groups A, B
$A \wedge B$—the semidirect product[7] of A, B.

We are interested in the classification of all the subgroups of G_0. Parts of this classification have been worked out in the literature in connection with various problems e.g. the 230 space groups in three dimensions, the space groups in one or two dimensions, the point groups, etc. The most important missing part is the classification of all the subgroups of G_0 which contain continuous rotations, and we present it in Sec. 2. In Sec. 3, we use the results obtained in Sec. 2 to present the complete classification of all the subgroups of G_0. In Sec. 4 we summarize the main results.

2 THE SUBGROUPS OF G_0 WHICH CONTAIN CONTINUOUS ROTATIONS

In this section we list all the subgroups of G_0 which contain continuous rotations, and may contain discrete rotational elements as well. These groups may be classified into three types:

1) The subgroups which contain translations and proper rotations only, but do not contain reflections and the inversion.

2) The groups which do not contain the inversion but do contain improper rotations, such as reflections, etc.

3) The groups containing the inversion itself.

Let us consider each type separately. The groups of the first type do not contain the inversion or products of the inversion with other elements. They are, therefore, subgroups of the (non-centrosymmetric) Euclidean group G. In order to find these groups, we consider first, the subgroups of G which contain only continuous elements. We shall then discuss the discrete rotational elements which can be added to each subgroup. The Euclidean group,

G, is a semidirect product of the three-dimensional translation group (P^3) and the three-dimensional rotation group (R^3):

$$G = R^3 \wedge P^3.$$

This is a lie group which has six generators:[6] L_i, P_i $i = 1, 2, 3$ which obey the following commutation relations:

$$[L_l, L_j] = i\varepsilon_{ljk}L_k,$$
$$[P_l, P_j] = 0,$$
$$[L_l, P_j] = i\varepsilon_{ljk}P_k.$$

where ε_{ljk} is the antisymmetric tensor of third rank. The subscripts 1, 2, 3 stand for x, y, z respectively. Any continuous subgroup of G is given by a set of operators which are linear combinations of the six generators of G. A necessary and sufficient condition for a set of operators to form a group is that it should be closed under the commutation relation, which means that the commutation relation of any two operators of the set must be a linear combination of the operators of this set. Using this condition it is found that there exist eleven continuous subgroups of G (including G itself) and they are given in Table I.[8]

TABLE I

The subgroups of G which contain *only* continuous elements.

No.	Group	No.	Group
1	P_x	7	$L_xP_xP_yP_z$
2	P_xP_y	8	$(L_x + \alpha P_x)$
3	$P_xP_yP_z$	9	$(L_x + \alpha P_x)P_yP_z$
4	L_x	10	L_xL_y
5	L_xP_x	11	$L_xL_yP_xP_yP_z$
6	$L_xP_yP_z$		

The subgroups of G which in addition to the continuous rotational elements also contain discrete ones, can be deduced by looking for all the possibilities of adding discrete rotational elements to the groups listed in Table I. There are two types of such elements which can be added to these groups:

a) For the uniaxial groups, i.e. groups containing one continuous rotational axis or screw axis, one may add two-fold rotation axes perpendicular to the continuous axis.

b) For groups containing a continuous screw axis $L_x + \alpha P_x$, one may add an n-fold rotation axis in the direction of the continuous axis. This is equivalent to introducing a discrete translation of $2\pi\alpha/n$ in this direction.

All possible groups obtained in this way are listed in Table II.

TABLE II

The subgroups of G which contain continuous rotations or screw axes (Type 1).

No.	Group	No.	Group
1	L_x	10	$(L_x + \alpha P_x)P_yP_z$
2	L_xP_x	11	$(L_x + \alpha P_x)2$
3	$L_xP_yP_z$	12	$(L_x + \alpha P_x)2P_yP_z$
4	$L_xP_xP_yP_z$	13	$(L_x + \alpha P_x)(n)_x$
5	L_x2	14	$(L_x + \alpha P_x)(n)_xP_yP_z$
6	L_x2P_x	15	$(L_x + \alpha P_x)2(n)_x$
7	$L_x2P_yP_z$	16	$(L_x + \alpha P_x)2(n)_xP_yP_z$
8	$L_x2P_xP_yP_z$	17	L_xL_y
9	$(L_x + \alpha P_x)$	18	$L_xL_yP_xP_yP_z$

Let us now turn to the subgroups of G_0 of the second type, i.e. the subgroups which contain improper rotations but do not contain the inversion itself. These groups may be obtained by finding all subgroups of index 2 for every group listed in Table II. It can be shown[9] that any subgroup of index 2 corresponds to a subgroup of G_0 which contains improper rotations but does not contain the inversion itself. The resulting groups are listed in the third column of Table III.

TABLE III

The subgroups of G_0 which contain continuous rotations or screw axes, and are not subgroups of G (types 2, 3).

Type 3				Type 2	
No.	Group	No.	Group	No.	Group
1	$L_x/m = (L_xI)$	7	L_x/mmP_yP_z	13	L_xm
2	L_x/mP_x	8	$L_x/mmP_xP_yP_z$	14	L_xmP_x
3	L_x/mP_yP_z	9	L_x/ma	15	$L_xmP_yP_z$
4	$L_x/mP_xP_yP_z$	10	L_x/maP_yP_z	16	$L_xmP_xP_yP_z$
5	L_x/mm	11	L_xL_yI	17	L_xa
6	L_x/mmP_x	12	$L_xL_yIP_xP_yP_z$	18	$L_xaP_yP_z$

The subgroups of G_0 which contain continuous rotations and the inversion itself are obtained by adding the inversion to the groups listed in Table II. The resulting groups are listed in the first two columns of Table III.

3 THE CLASSIFICATION OF THE SUBGROUPS OF G_0

The subgroups of G_0 may be divided into ten classes as listed in Table IV, according to their translation subgroup. In the following, we list the subgroups of G_0 which belong to each class. We also list the point group and site symmetries of the groups of classes 1 and 2.

TABLE IV

The ten classes of translation subgroups of the three dimensional translation group P^3

No.	Translation Group	Explanation	Physical Examples	Hermann's Classification
1	P^3	Three infinitesimal translations.	amorphous solids, powders liquids, nematics	SSS; SSD
2	P^2xT	One discrete translation (T) perpendicular to a plane of infinitesimal translations (P^2).	smectics, cholesterics	SSR; SS (RD) RDS; DDR
3	PxT^2	A plane of discrete translations (T^2) perpendicular to a continuous translation (P).		RRD; RD (RD)
4	T^3	Three discrete translations.	crystals	(RD) (RD) (RD)
5	P^2	A plane of infinitisimal translations.	liquid or nematic layer	
6	PxT	A discrete and infinitesimal translations perpendicular to each other.		
7	T^2	Two discrete translations.	crystalline surface	
8	P	One infinitesimal translation.		
9	T	One discrete translation.		
10	I	No translation symmetry.	molecules	

1 The groups which contain the three-dimensional continuous translation subgroup

The three-dimensional continuous translational group may appear with any one of the point groups (discrete, continuous, and groups which contain rotational elements of both kinds). Following Shubnikov,[10] we list in Table V all the possible point groups. In addition to the rotational elements of the point groups, the groups which belong to class 1 of Table IV contain three continuous translations, P_x, P_y, and P_z. Each group of this class has one kind of special position whose site symmetry (a subgroup which transforms at least one point onto itself) is equal to its point group. Substances having symmetries of this class are translationally invariant and all their points are therefore equivalent, and have a site symmetry equal to the point group. As a consequence, each point is a "special position" having a non-trivial symmetry (except for the group whose point group is trivial). For crystal symmetries (class 4), this is not the case, and any one of the 230 space groups contains "general position" points with trivial symmetry.

2 Groups whose translation subgroup contains two continuous and one discrete translations

These are the space groups in one dimension (the symmetry of a line perpendicular to the plane of continuous translations). The space groups in one dimension which contain an n-fold rotation axis where $n = 1, 2, 3, 4, 6$ were given by Belov.[11] In Tables VI–IX we list all possible space groups in one dimension. The groups which belong to this class contain, in addition to the elements which appear in the tables, two continuous translations P_y, P_z and a discrete translation T_x. In Table VI we list the groups which contain an n-fold rotation axis where $n = 1$, 2. In Tables VII and VIII we list the groups which contain a rotation axis of even or odd order larger than 2, respectively. In Table IX we list the groups with a continuous rotation axis or a screw axis. This table is based on Tables I–III derived in Sec. 2.

3 The groups whose translation subgroup contains two discrete translations perpendicular to a continuous translation

These are the 80 space groups in two dimensions (the plane of the discrete translations) and they are listed in Table X.[12] The groups belonging to this class contain a continuous translation P_z in addition to elements listed in the Table.

TABLE V

Point groups

I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV	XV	XVI
1	2	$\bar{1}$	$\bar{4}$	$\bar{2}$	$2/m$	$2mm$	$1m$	222	12	$2/mmm$	$\bar{2}m2$	$\bar{4}2m$	$\bar{1}m$	23	$3m$
3	4	$\bar{3}$	$\bar{8}$	$\bar{6}$	$4/m$	$4mm$	$3m$	422	32	$4/mmm$	$\bar{6}m2$	$\bar{8}2m$	$\bar{3}m$	43	$43m$
5	6	$\bar{5}$	$\overline{12}$	$\overline{10}$	$6/m$	$6mm$	$5m$	622	52	$6/mmm$	$\overline{10}m2$	$\overline{12}2m$	$\bar{5}m$	53	$3m3$
7	8	$\bar{7}$	$\overline{16}$	$\overline{14}$	$8/m$	$8mm$	$7m$	822	72	$8/mmm$	$\overline{14}m2$	$\overline{16}2m$	$\bar{7}m$		$53m$
L_x		L_x/m				L_xm		L_x2		L_x/mm				L_xL_y	L_xL_yI

TABLE VI

The space groups in one dimension with 1- or 2-fold rotation axes, or screw axes.

No.	Space group	Point group	Site symmetries
1	111	1	1
2	$\bar{1}$	$\bar{1}$	$\bar{1}$; 1
3	211	211	211
4	$2_1 11$	211	1
5	121	121	121; 1
6	222	222	222; 211
7	$2_1 22$	222	121; 112; 1
8	*m*11	*m*11	*m*11; 1
9	2/*m*11	2/*m*11	2/*m*11; 211
10	$2_1/m11$	2/*m*11	*m*11; 1
11	1*m*1	1*m*1	1*m*1
12	12/*m*1	12/*m*1	12/*m*1; 1*m*1
13	1*a*1	1*m*1	1
14	12/*a*1	12/*m*1	121; 1
15	*mm*2	*mm*2	*mm*2; 1*m*1
16	*ma*2	*mm*2	*m*11; 112; 1
17	2*mm*	2*mm*	2*mm*
18	$2_1 am$	2*mm*	11*m*
19	2*aa*	2*mm*	211
20	*mmm*	*mmm*	*mmm*; 2*mm*
21	*mam*	*mmm*	*m*2*m*; 11*m*
22	*maa*	*mmm*	*m*11; 222; 121; 112; 1

TABLE VII

The space groups in one dimension which contain (2*n*) fold rotation axis, or screw axis, $2n > 2$.

No.	Space group	Point group	Site symmetries
1	(2*n*)	(2*n*)	(2*n*)
2†	$(2n)_k$	(2*n*)	(*l*)
3	$(\overline{2n})$	$(\overline{2n})$	$(\overline{2n})$; (*n*)
4	(2*n*)/*m*	(2*n*)/*m*	(2*n*)/*m*; (2*n*)
5	$(2n)_n/m$	(2*n*)/*m*	(*n*)/*m*; (*n*)
6	(2*n*)22	(2*n*)22	(2*n*)22; (2*n*)
7†	$(2n)_k 22$	(2*n*)22	(*l*)22; 2; 1
8	$(\overline{2n})2m$	$(\overline{2n})2m$	$(\overline{2n})2m$; (*n*)*mm*
9	$(\overline{2n})2a$	$(\overline{2n})2m$	$(\overline{2n})$; (*n*)22; (*n*)
10	(2*n*)*mm*	(2*n*)*mm*	(2*n*)*mm*
11	$(2n)_n ma$	(2*n*)*mm*	(*n*)*mm*
12	(2*n*)*aa*	(2*n*)*mm*	(2*n*)
13	(2*n*)/*mmm*	(2*n*)/*mmm*	(2*n*)/*mmm*; (2*n*)*mm*
14	$(2n)_n/mma$	(2*n*)/*mmm*	(*n*)/*mmm*; (*n*)*mm*
15	(2*n*)/*maa*	(2*n*)/*mmm*	(2*n*)/*m*; (2*n*)

† $l = (2n) \cdot k/S(2n, k)$, where $S(2n, k)$ is the least common multiplier of 2*n* and *k*. See notations.

TABLE VIII

The space groups in one dimension which contain $(2n + 1)$ fold rotation axis, or screw axis, $2n + 1 > 1$.

No.	Space group	Point group	Site symmetries
1	$(2n + 1)$	$(2n + 1)$	$(2n + 1)$
2†	$(2n + 1)_k$	$(2n + 1)$	(l)
3	$(\overline{2n + 1})$	$(\overline{2n + 1})$	$(\overline{2n + 1})$; $(2n + 1)$
4	$(2n + 1)m$	$(2n + 1)m$	$(2n + 1)m$
5	$(2n + 1)a$	$(2n + 1)m$	$(2n + 1)$
6	$(2n + 1)2$	$(2n + 1)2$	$(2n + 1)2$: $(2n + 1)$
7†	$(2n + 1)_k2$	$(2n + 1)2$	$(l)2$ (odd l) or $(l)22$ (even l); (l)
8	$(\overline{2n + 1})m$	$(\overline{2n + 1})m$	$(\overline{2n + 1})m$; $(2n + 1)m$
9	$(\overline{2n + 1})a$	$(\overline{2n + 1})m$	$(\overline{2n + 1})$; $(2n + 1)$

† $l = (2n + 1) \cdot k/S(2n + 1, k)$, where $S(2n + 1, k)$ is the least common multiplier of $2n + 1$ and k. See notations.

TABLE IX

The space groups in one dimension which contain continuous rotation axis or screw axis.

No.	Space group	Point group	Site symmetries
1	L_x	L_x	L_x
2	L_x2	L_x2	L_x2; L_x
3	L_xm	L_xm	L_xm
4	L_xa	L_xm	L_x
5	L_x/m	L_x/m	L_x/m; L_x
6	L_x/mm	L_x/mm	L_x/mm; L_xm
7	L_x/ma	L_x/mm	L_x/m; L_x
8	$(L_x + \alpha P_x)$	L_x	1
9	$(L_x + \alpha P_x)2$	L_x2	2
10	$(L_x + \alpha P_x)(n)_x$	L_x	$(n)_x$
11	$(L_x + \alpha P_x)(n)_x2$	L_x2	$(n)_x22$

4 The groups which contain a three-dimensional discrete translation subgroup

These are the 230 space groups. They are listed in the International Tables of X-ray Crystallography.[13]

The groups which belong to classes 1–4 contain a three-dimensional translation subgroup. These groups may describe the symmetry of liquids, amorphous solids, powders, liquid crystals and crystals. We turn now to groups which contain a two-dimensional translation subgroup. These groups may describe the symmetry of surfaces.

TABLE X

The space groups in two dimensions.

No.	Group	No.	Group	No.	Group	No.	Group
1	$P\bar{1}$	21	$C2/m11$	41	$P312$	61	$P31m$
2	$P11b$	22	$C222$	42	$P321$	62	$P6$
3	$P112/b$	23	$Pmmb$	43	$P\bar{3}$	63	$P6mm$
4	$P121$	24	$Pmab$	44	$P\bar{3}12/m$	64	$P11m$
5	$P12_11$	25	$Pbmb$	45	$P\bar{3}2/m1$	65	$P112/m$
6	$C121$	26	$Pbab$	46	$P622$	66	$Pm2m$
7	$Pm2_1b$	27	$Pmmm$	47	$P1$	67	$Pb2_1m$
8	$Pb2b$	28	$Pman$	48	$P112$	68	$Cm2m$
9	$P2mb$	29	$Pban$	49	$Pm11$	69	$Pmmm$
10	$P2_1ab$	30	$Cmma$	50	$P1a1$	70	$Pmam$
11	$Pm2_1n$	31	$P\bar{4}$	51	$Cm11$	71	$Pbam$
12	$Pb2n$	32	$P4/n$	52	$Pmm2$	72	$Cmmm$
13	$Cm2a$	33	$P\bar{4}m2$	53	$Pma2$	73	$P4/m$
14	$P2/m11$	34	$P\bar{4}2m$	54	$Pba2$	74	$P4/mmm$
15	$P222$	35	$P422$	55	$Cmm2$	75	$P4/mbm$
16	$P12/a1$	36	$P\bar{4}b2$	56	$P4$	76	$P3/m$
17	$P2_1/m11$	37	$P\bar{4}2_1m$	57	$P4mm$	77	$P3/mm2$
18	$P2_1/b11$	38	$P42_12$	58	$P4bm$	78	$P3/m2m$
19	$P2_12_12$	39	$P4/nmm$	59	$P3$	79	$P6/m$
20	$P2_122$	40	$P4/nbm$	60	$P3m$	80	$P6/mmm$

5 Groups which contain two-dimensional continuous translation subgroup

The two continuous translations P_x, P_y may appear with any one of the uniaxial point groups. These groups are listed in columns 1–14 of Table V, where the symmetry axis is in the z-direction. In addition to the groups appearing in the Table, there are 4 more groups which belong to this class and contain symmetry axes of order $n = 1, 2$ in the x-y plane. These are (see notations):

$$2_x;\ m_x;\ 2_x/m;\ 2_xmm$$

6 Groups which contain one discrete and one continuous translations perpendicular to each other

These are the space groups of a line, (the line of the discrete translation) in a plane (the plane of the two translations). There are 31 groups of this kind and they were given by Belov.[11] We list them in Table XI where the discrete and continuous translations are in direction x, y, respectively.

TABLE XI

The groups which contain a discrete translation (x axis) and a continuous one (y axis).

No.	Group	No.	Group	No.	Group
1	111	11	$2_1/m11$	21	$ma2$
2	$\bar{1}$	12	$1m1$	22	$m2m$
3	211	13	$1\ 2/m\ 1$	23	$m2a$
4	$2_1 11$	14	$1a1$	24	$2mm$
5	121	15	$1\ 2/a\ 1$	25	$2_1 am$
6	112	16	$11m$	26	$2_1 ma$
7	222	17	$11a$	27	$2aa$
8	$2_1 22$	18	$11\ 2/m$	28	mmm
9	$m11$	19	$11\ 2/a$	29	mam
10	$2/m11$	20	$mm2$	30	mma
				31	maa

7 The groups which contain two discrete translations

These are the 80 space groups in two dimensions and they are listed in Table X.[12]

The groups which describe the symmetry of a line are given by the two following classes.

8 The groups which contain one continuous translation

The continuous translation P_z may appear with any one of the uniaxial point groups. These are listed in columns 1–14 of Table V. In addition to the groups appearing in this Table there are four more groups which belong to this class, and contain symmetry axes of order $n = 1, 2$ in the x-y plane. These are:

$$2_x;\ m_x;\ 2_x/m;\ 2_x mm$$

9 The groups which contain one discrete translation

These are the space groups in one dimension and they are listed in Tables VI–IX. The site symmetries given in these Tables are irrelevant. In addition to the elements appearing in these Tables, each group contains a discrete translation T in x-direction (see explanations to the groups of class 2).

10 Groups which do not contain translational elements

These are the point groups listed in Table V.

4 CONCLUDING REMARKS

In this work we presented a classification of all possible symmetry groups of rigid transformations. We made no attempt to give a general definition of the symmetry of a given phase, and we hope to refer to this important problem in a later publication. For practical purposes, we shall define the symmetry of the liquid crystals as the symmetry group of the order parameter which is used to describe them. The following groups are obtained for the known liquid crystals:

1) nematics (uniaxial) – $L_x/mm\ P_x P_y P_z$,
(Table V columns XI–XIII).
2) Cholesterics – $(L_x + \alpha P_x)2\ 2_x P_y P_z$,
(Table IX group 11).
3) Smectic A – $L_x/mm\ T_x P_y P_z$,
(Table IX Group 6).
4) Smectic C – $\bar{1}\ T_x P_y P_z$,
(Table VI Group 2).

The structure of the more exotic smectics[14] (B, G, E, H) has not been determined yet. For example, there exist two different models which might describe the smectic B:[15]

1) The smectic B has a three-dimensional order. In this case it is in fact a crystal, and it is described by one of the 230 space groups (class 4).

2) The smectic B consists of two-dimensional ordered layers which slip freely on each other. The order parameter, in this case, is not clear.

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Notations

We use the conventional International symbols for the crystallographic groups.[13] A straightforward generalization is used for the noncrystallographic groups.

$(n)_k$—an n-fold screw axis. The translation accompanying the rotation is $k/n \cdot a$, where a is the smallest translation in the direction of the rotation axis.
$(n)_x$—an n-fold rotation axis in the x direction.
2_x—two-fold rotation axis in the x direction.
I, or $\bar{1}$—inversion

L_x, L_y, L_z— continuous rotation elements along the x, y, z axes, respectively.

P_x, P_y, P_z— continuous translation elements in the x, y, z directions, respectively.

T_x, T_y, T_z— discrete translation elements in the x, y, z directions, respectively.

$(L_x + \alpha P_x)$— a continuous screw axis in the x-direction, with a pitch of $2\pi\alpha$.

L_x/m— a continuous rotation axis in the x direction, and a mirror plane perpendicular to it.

$L_x m$— a continuous rotation axis in the x direction, and a mirror plane containing the rotation axis.

$L_x 2$— a continuous rotation axis in the x-direction, and a two-fold rotation axis perpendicular to it.

$(L_x + \alpha P_x)(n)_x$— a continuous screw axis, and an n-fold rotation axis both in the x-direction.

$L_x a$— a continuous rotation axis in the x-direction, and a glide plane containing the rotation axis.

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2. A. J. Mabis, *Acta Cryst.* **15**, 1152 (1962).
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4. By centrosymmetric liquid we mean a liquid which consists of equal parts of right and left handed molecules, or a liquid whose molecules are equal to their mirror image.
5. L. D. Landau, and M. E. Lifshitz, *Statistical Physics* (Pergamon, New York, 1968) 2nd Ed. p. 404.
6. W. Miller, *Lie Theory and Special Functions* (Academic Press, New York 1968) p. 255.
7. J. S. Lomont, *Applications of Finite Groups* (Academic Press, New York 1959) p. 29.
8. The requirement that the commutator of any two operators of one set should be a linear combination of the operators of this set apparently generates groups other than those appearing in Table I, e.g. the group whose generator is $L_x + \alpha P_y$. It can be shown that any other group not listed in Table I is equivalent to some group in the list. Specifically, the group $L_x + \alpha P_y$ is equivalent to the group L_x, with the rotation axis shifted by an amount α along the z axis. This follows from the relation: $L_x + \alpha P_y = e^{i\alpha P_z} L_x e^{-i\alpha P_z}$.
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